1 NUMBER THEORY

The concepts discussed in this lecture will be your first step towards a general understanding of the requirements of Maths for entrance exam. As you proceed with this lecture, you will find that many concepts given here have already been taught to you in the elementary classes at school level. This would further help build confidence in you.

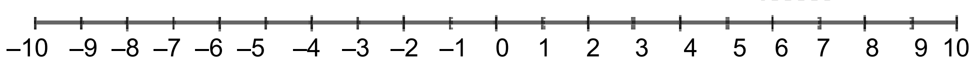
Understanding Numbers

Numbers

A measurement carried out, of any quantity, leads to a meaningful value called the ***number***. This value may be positive or negative depending on the direction of the measurement and can be represented on the number line.

Number line:

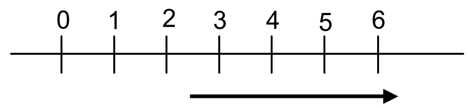
The number line is a line labelled with the numbers in increasing order from left to right, that extends in both directions:



For any two different places on the number line, the number on the right is greater than the number on the left.

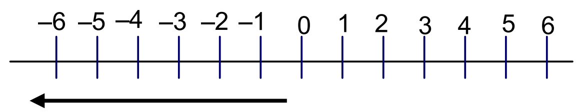
Positive Number:

Any number greater than zero is a positive number. i.e. numbers on the right hand side of zero on the number line.



Negative Number:

Any number less than zero is a negative number. i.e. numbers on the left hand side of zero on the number line.



Types of Numbers

1. Real and 2. Imaginary

Real Numbers:

Real numbers are those which can represent actual physical quantities in a meaningful way e.g. length, height, density etc.

Examples: $1, 2, \frac{1}{5}, -\frac{1}{8}$ etc.

Imaginary numbers:

All numbers which are not real, are imaginary (complex) numbers.

Examples: $\sqrt{-1}$, $\sqrt{-7}$ etc.

Real Numbers

Real numbers are basically of two types:

Rational Numbers:

Rational Number is defined as the ratio of two integers i.e. a number that can be represented by a fraction of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Examples: Finite decimal numbers, whole numbers, integers, fractions i.e.

$\frac{3}{5}, \frac{16}{9}, 0.666\ldots\infty = \frac{2}{3}$ etc.

Irrational numbers:

Any number which can not be represented in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$ is an irrational number. *An infinite non-recurring decimal is an irrational number.*

Examples: π, $\sqrt{5}$, $\sqrt{7}$.

Integers:

The set of Integers; I = {0, ± 1, ± 2, ± 3, …}

Whole Numbers:

The numbers 0, 1, 2, 3, 4, …∞, W = {0, 1, 2, 3….}

Natural Numbers:

The numbers 1, 2, 3, 4, 5…are known as natural numbers. The set of natural numbers is denoted by N. Hence, N = {1, 2, 3, 4…}

> ***Do you know?***
>
> By definition 0 is an even number.

Even Numbers:

The numbers divisible by 2 are even numbers. Eg, 2, 4, 6, 8, 10 …Even numbers are expressible in the form 2n, where n is an integer. Thus – 2, – 6, etc. are also even numbers.

Odd Numbers:

The numbers not divisible by 2 are odd numbers e.g. 1, 3, 5, 7, 9…Odd numbers are expressible in the form (2n + 1) where n is an integer (not necessarily prime). Thus. – 1, – 3, – 9 etc. are all odd numbers.

> **TIP**
>
> If a number has no prime factor equal to or less than its square root, then the number is prime.

Prime Numbers:

A natural number that has no other factors besides itself and unity is a prime number.
Examples: 2, 3, 5, 7, 11, 13, 17, 19 …

Composite Numbers:

A composite number has other factors besides itself and unity. e.g. 8, 72, 39, etc.

Perfect Numbers:

A number is said to be a perfect number if the sum of ALL its factors excluding itself (but including 1) is equal to the number itself.

Example: 6 is a perfect number because the factors of 6, i.e., 1, 2 and 3 add up to the number 6 itself.
Other examples of perfect numbers are 28, 496, 8128, etc. There are 27 perfect numbers discovered so far.

Fractions

A fraction denotes part or parts of a unit. Several types are:

1. **Common Fraction:** Fractions whose denominator is not 10 or a multiple of it. e.g. $\frac{2}{3}, \frac{17}{18}$ etc.
2. **Decimal Fraction:** Fractions whose denominator is 10 or a multiple of 10.
3. **Proper Fraction:** In this the numerator < denominator e.g. $\frac{2}{10}, \frac{6}{7}, \frac{8}{9}$ etc.

   Hence its value < 1.
4. **Improper Fraction:** In these the numerator > denominator e.g. $\frac{10}{2}, \frac{7}{6}, \frac{8}{7}$ etc.

   Hence its value > 1.

5. **Mixed Fractions**: When a improper fraction is written as a whole number and proper fraction it is called mixed fraction. e.g. $\frac{7}{3}$ can be written as $2 + \frac{1}{3} = 2\frac{1}{3}$.

Absolute Value of a Number

The absolute value of a number |a| is a and is always positive. By definition

$|a| = a$ if a = + ve & a = – a if a = – ve

$|a|$ is read as MODULUS of a or **Mod a**

Example: |79| = 79 or |–98| = –(–98) etc.

Divisibility Test

No.	Rule	Example/s
2	Last digit of number is even.	12**8**, 14**6**, 3**4** etc.
3	Sum of digits of a given number is divisible by 3	102, 192, 99 etc.
4	Number formed by last two right hand digits of given number is divisible by 4	5**76**, 1**44** etc.
5	Last digit is either five or zero	11115**5**, 397**0**, 14**5** etc.
6	Number is divisible by both 2 and 3	714, 509796, 1728 etc.
7	Take the ones digit off of n, double it and subtract the doubled number from the remaining number. If the result is evenly divisible by 7 (e.g. 14, 7, 0, – 7, etc.), then the number is divisible by seven. This process may need to be repeated several times.	3101. Take off ones: 310. Subtract from this ones doubled: 310-2: 308. Take off ones: 30. Subtract the ones double, which is 16: 14. 14 is divisible by 7, so 3101 is too.
7	The rule which holds good for numbers with more than 3 digits is as follows: (A) Group the numbers in three from unit digit (B) Add the odd groups and even groups separately (C) The difference of the odd and even groups should be 0 or divisible by 7	Let's take 85437954 . The groups are 85, 437, 954 Sum of odd groups = 954 + 85 = 1039 Sum of even groups = 437 Difference = 602. Which is divisible by 7, hence the number is divisible by 7.
8	Number formed by the last three right hand digits of a number is divisible by '8'.	**512**, 4**096**, 1**304** etc.

9	Sum of its digits is divisible by 9.	1287, 11583, 2304 etc.
10	Unit digit is zero.	100, 170, 10590 etc.
11	When the difference between the sums of digits in the odd and even places is either zero or a multiple of 11.	e.g.17259 Sum of digits in even places = 7 + 5 = 12, Sum of digits in the odd places = 1 + 2 + 9 = 12 Hence 12 – 12 = 0.
12	It is divisible by 3 & 4 both.	672, 8064 etc.
13	The rule which holds good for numbers with more than 3 digits is as follows: (A) Group the numbers in three from unit digit (B) Add the odd groups and even groups separately (C) The difference of the odd and even groups should be 0 or divisible by 13	Let's take 35250799415 The groups are 035, 250, 799, 415 Sum of odd groups = 035 + 799 = 834 Sum of even groups = 250 + 415 = 665 Difference = 834 – 615 =169 which is divisible by 13, hence the number is divisible by 13.
25	When the number formed by last two right hand digits is divisible by 25.	10**<u>25</u>**, 34**<u>75</u>**, 555**<u>50</u>** etc.
125	When the number formed by last three right hand digits is divisible by 125.	2**<u>125</u>**, 4**<u>250</u>**, 6**<u>375</u>** etc.

Do You know ?

1. When any number with even number of digits is added to its reverse, the sum is always divisible by 11.

e.g. 2341 + 1432 = 3773, which is divisible by 11.

2. If X is a prime number then for any whole number "a" (a^X – a) is divisible by X

e.g. Let X = 3 and a = 5. Then according to our rule 5^3 – 5 should be divisible by 3.

Now (5^3 – 5) = 120 which is divisible by 3.

Recurring Decimals

If in a decimal fraction a figure or a set of figures is repeated continually, then such a number is called a ***recurring decimal***.

If a single figure is repeated, then it is shown by putting a dot on it. Also, if a set of figures is repeated, we express it by putting one dot at the starting digit and one dot at the last digit of the repeated set.

Examples: (i) $\frac{2}{3} = 0.666\ldots = 0.\dot{6} = 0.\overline{6}$

(ii) $\frac{22}{7} = 3.142857142857\ldots = 3.\dot{1}4285\dot{7}$ or $3.\overline{142857}$

(iii) $\frac{17}{90} = 0.1\overline{8}$

Pure Recurring Decimals:

A decimal in which all the figures after the decimal point repeat, is called a ***pure recurring decimal.***

Examples: $0.\dot{6}$, $0.\overline{6}$ are examples of pure recurring decimals.

Mixed Recurring Decimals:

A decimal in which some figures do not repeat and some of them are repeated is called a ***mixed recurring decimal.***

Example $0.1\overline{8}$ is a mixed recurring decimal.

Converting Recurring Decimal into Fraction

1. PURE RECURRING DECIMAL

Example

TIP

Write the recurring figures only once in the numerator and take as many nines in the denominator as the number of repeating figures.

Convert $0.\overline{6}$ to a rational number

Sol. Let p = $.\overline{6}$ …(i)

The Bar is over 1 numeral

So multiply both side by 10^1 & we get

$10p = 6.\overline{6}$ …(ii)

Subtracting (1) from (2) we get

9p = 6

or $p = \frac{6}{9} = \frac{2}{3}$

Similarly 1) $0.\dot{2}3\dot{4}$ = 234/999. 2) $3.\dot{5} = 3\frac{5}{9}$. 3) $2.\dot{0}3\dot{5} = 2\frac{35}{999}$.

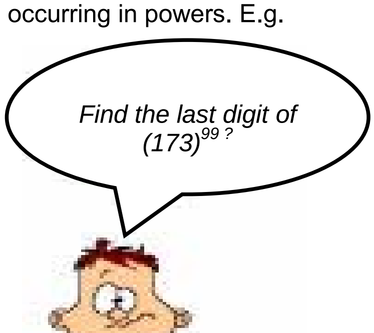
2. MIXED RECURRING DECIMAL

RULE: In the numerator take the difference between the number formed by all digits after decimal point (taking repeated digits only once) and that formed by the digits which are not repeated. In the denominator, take the number formed by as many nines as there are repeating digits followed by as many zeros as is the number of non-repeating digits.

Ex. $2.53\dot{6} = 2\left(\frac{536-53}{900}\right) = 2\frac{161}{300}$.

Finding the Unit Digit

At times there are questions that require the students to find the units digit in case of the numbers occurring in powers. E.g.



Seems to be difficult. But it's very simple if we understand that **the units digit of a product is determined by whatever is the digit at the units place irrespective of the number of digits**. E.g. 5 × 5 ends in 5 & 625 × 625 also ends in 5.

Now let's examine the pattern that a number generates when it occurs in powers of itself.
See the last digit in powers of 2.

Power of 2	Last Digit	Power of 2	Last Digit	Power of 2	Last Digit
2^1	2	2^5	2	2^9	2
2^2	4	2^6	4	2^{10}	4
2^3	8	2^7	8	2^{11}	8
2^4	6	2^8	6	2^{12}	6

So we note that last digit repeats itself after 4 operations. This forms the basis of finding last digits. Since we have only 10 possibilities at the units place in the decimal number system, we analyze the pattern for the same & arrive at the following conclusions:

Number	Repeating Series	Cycle of repetition
0	0	1
1	1	1
2	2, 4, 8, 16	4
3	3, 9, 27, 81	4
4	4, 16	2
5	5	1
6	6	1
7	7, 49. 343, 2301	4
8	8, 64 512, 4096	4
9	9, 81	2

Important Points

Following points need to be observed for the numbers occurring at the units place:

- The nos. 1, 0, 5, 6 stay as they are i.e. even if we raise them 'n' times the digit at the units place will remain same.
- Two nos. 4 & 9 have a cycle time of 2. Thus;

Number	Power	Unit's Digit
4	Odd/Even	4(Remains same)/6
9	Odd/Even	9(Remains same)/1

- Remaining four numbers (2, 3, 7 & 8) have a cycle time of 4 each.
- Every fifth power in the case of nos. having a cycle time other than 1 is the number itself. Or in other words, a group of 4 nos. does not affect the number that occurs at the units place.

TIP

So the rule for the numbers ending in 2, 3, 7 & 8 is raise the units digit of the given number to the remainder that we get after dividing the power by 4.

Let us consider an example: ***Find the last digit of $(173)^{99}$.***

Sol. We notice that the exponent is 99. On dividing 99 by 4 we get 24 as the quotient & 3 as the remainder. Now these 24 pairs of 4 each do not affect the no. at the units place So, $(173)^{99} \approx (173)^3$. Now, the number at the units place is $3^3 = 2\underline{\mathbf{7}}$.

What happens if the exponent turns out to be a perfect multiple of 4. In these cases the number at the units place is always the 4^{th} power & not 1 as one might be tempted to think. Let us again consider another example: Find the last digit of $(172)^{96}$. In this case on dividing by 4 we get 0 as the remainder. Now since anything raise to the power 0 is 1, one is tempted to think of the answer as 1.But here it is important to see that we have 24 pairs of 2^4 & each pair will have 6 as the units digit & 6 raised 24 times will still give us 6.

Thus, steps for finding the number at the units place can be listed as under (only for the nos. that change i.e. 2, 3, 4, 7, 8 & 9):

1. Divide the exponent /power by 4.
2. Any remainder other than 0 will result in the units digit being that power of the no. at the units place.
3. If the remainder turns out to be zero, the units digit is the 4^{th} power of the no. at the units place.

Factors

A factor is a number that divides another number completely. e.g. Factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24.

Number of Factors:

$N = p^a \times q^b \times r^c$

Where p, q, and r are prime numbers and a, b, and c are the no. of times each prime number occurs , then the number of factors of n is found by (a + 1) (b + 1)(c + 1).

Find the number of factors of $2^4 \times 3^2$.

(4 + 1) (2 + 1) = 5(3) = 15

Number of Ways of Expressing a Given Number as a Product of Two Factors:

The given number N (which can be written as equal to $a^p\ b^q\ c^r$…where a, b, c are prime factors of N and p, q, r…. are positive integers) can be expressed as the product of two factors in different ways.

The number of ways in which this can be done is given by the expression

$$\tfrac{1}{2}\{(p + 1)\ (q + 1)\ (r + 1)\ldots.\}$$

So, 148 can be expressed as a product of two factors in $\frac{6}{2}$ or 3 ways.

{Because (p + 1) (q + 1) (r + 1) in the case of 148 is equal to 6}.

If p, q, r etc. are all even, then the product (p + 1) (q + 1) (r + 1)…. becomes odd and the above rule will not be valid since we cannot take ½ of an odd number to get the number of ways, if p, q, r …. are all even, it means that the number N is a perfect square. This situation arises in the specific cases of perfect squares because a perfect square can also be written as {square root x square root}. So, two different cases arise in case of perfect squares depending on whether we would like to consider the writing of the number as {square root x square root} as one way or not.

Thus, to find out the number of ways in which a perfect square can be expressed as a product of 2 factors, we have the following two rules:

(1) As a product of two DIFFERENT factors in ½ **{(p + 1) (q + 1) (r + 1)…. – 1}** ways (excluding $\sqrt{N} X \sqrt{N}$).

(2) As a product of two factors (including $\sqrt{N} X \sqrt{N}$) in ½ **{(p + 1)(q + 1)(r + 1)…+1}** ways.

Find the number of factors the number 1008 has.

Sol. First, express 1008 as a product of its prime factor (Note that to express a given number as a product of its prime factor, we first need to identify the prime factors the given number has by applying the rules for divisibility that have already been discussed).

$1008 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 = 2^4 \times 3^2 \times 7^1$

Hence, the number of factors 1008 has is (4 + 1) (2 + 1) (1 + 1) which is equal to 30.

How many divisors does the number 8580 have excluding 1 and itself?

Sol. Note that the two terms of factors and divisors are used interchangeably.

First, express 8580 in terms of its prime numbers.

We can clearly see that 3, 4 and 5 are factors of this number.

We can also see that number is divisible by 11.

$8580 = 2 \times 2 \times 3 \times 5 \times 11 \times 13 = 2^2 \times 3^1 \times 5^1 \times 11^1 \times 13^1$.

Hence, the number 8580 has (2 + 1) (1 + 1) (1 + 1) (1 + 1) (1 + 1) i.e., 48 factors.

Excluding 1 and itself, the number has 46 factors (48 – 2).

In how many ways can the number 13260 be written as the product of two factors?

Sol. We can see that 13260 is divisible by 3, 4 and 5.

So, we can write 13260 as $3 \times 4 \times 5 \times 221$.

Now checking out for the factors of 221, we find that it is equal to 13×17.

Hence, $13260 = 2^2 \times 3^1 \times 5^1 \times 13^1 \times 17^1$

So, 13260 can be written as a product of two factors i.e. 1/2 {(2 + 1) (1 + 1) (1 + 1) (1 + 1) (1 + 1)}, i.e., 24 ways.

In how many ways can the number 7056 be written as a product of two different factors?

Sol. By observation, we can say that 7056 is divisible by 8 as well as 9.

So, 7056 can be written as equal to $8 \times 2 \times 9 \times 49$.

Thus, $7056 = 2^4 \times 3^2 \times 7^2$.

Since all the powers here are even, the given number is a perfect square.

The question has asked us to find the number of ways of writing the number as a product of "two different" factors, we cannot consider (square root x square root). So, the required number of ways are ½ {(4 + 1) (2 + 1) (2 + 1) – 1} = ½ {45 – 1} = 22.

Factorial

TIP

0! is equal to 1.

Factorial is defined for any positive integer. It is denoted by ⌊ or !. Thus "Factorial n" is written as n! or ⌊n . n ! is defined as the product of all the integers from 1 to n.
Thus n! = 1.2.3. …. n (n – 1).

FINDING THE LARGEST POWER OF A NUMBER CONTAINED IN THE FACTORIAL OF A GIVEN NUMBER

Find the largest power of 3 that can divide 95! without leaving any remainder.

OR

Find the largest power of 3 contained in 95!.

Sol. First look at the detailed explanation and then look at a simpler method for solving the problem.

When we write 95! in its full form, we have 95 × 94 × 93 ….. × 3 × 2 × 1. When we divide 95! by a power 3, we have these 95 numbers in the numerator. The denominator will have all 3's. The 95 numbers in the numerator have 31 multiplies of 3 which are 3, 6, 9…. 90, 93.Corresponding to each of these multiplies we can have a 3 in the denominator which will divide the numerator completely without leaving any remainder, i.e. 3^{31} can definitely divide 95!

Further every multiple of 9, i.e. 9, 18, 27, etc. after canceling out a 3 above, will still have one more 3 left. Hence for every multiple of 9 in the numerator, we an have an additional 3 in the denominator. There are 10 multiples of 9 in 95 i.e. 9, 18….81, 90. So we can take 10 more 3's in the denominator.

Similarly, for every multiple of 3^3 we can take an additional 3 in the denominator.
Since there are 3 multiples of 27 in 91 (they are 27, 54 and 81), we can have three more 3's in the denominator.

Next, corresponding to every multiple of 3^4 i.e. 81 we can have one more 3 in the denominator. Since there is one multiple of 81 in 95, we can have one additional 3 in the denominator.

Hence the total number of 3's we can have in the denominator is 31 + 10 + 3 + 1, i.e., 45. So 3^{45} is the largest power of 3 that can divide 95! without leaving any remainder.

The same can be done in the following manner also.
Divide 95 by 3 you get a quotient of 31. Divide this 31 by 3 we get a quotient of 10. Divide this 10 by 3 we get a quotient of 3. Divide this quotient of 3 once again by 3 we get a quotient of 1. Since we cannot divide the quotient any more by 3 we stop here. Add all the quotients, i.e. 31 + 10 + 3 + 1 which gives 45 which is the highest power of 3.

3	95
3	31 ---> Quotient
3	10 ---> Quotient
3	3 ---> Quotient
	1 ---> Quotient

Add all the quotients 31 + 10 + 3 + 1, which give 45.
{Note that this type of a division where the quotient of one step is taken as the dividend in the subsequent step is called ***"Successive Division"***. In general, in successive division, the divisor need not be the same (as it is here). Here, the number 95 is being successively divided by 3.

Please note that this method is applicable only if the number whose largest power is to be found out is ***a prime number***.
If the number is not a prime number, then we have to write the number as the product of relative primes, find the largest power of each of the factors separately first. Then the smallest, among the largest powers of all these relative factors of the given number will give the largest power required.

Example

Find the largest power of 5 contained in 263!

Sol. Divide 263 successively by 5 (since 5 is a prime number)

5	263
5	52 ---> Quotient
5	10 ---> Quotient
5	2 ---> Quotient

Since 2 cannot be divided by 5 anymore, we stop here and all the quotients 52 + 10 + 2 to give 64. Hence 64 is the largest power of 5 that can divide 263! without leaving any remainder.

Example

Find the largest power of 12 that can divide 200!

Sol. Here we cannot apply Successive Division method because 12 is not a prime number. Resolve 12 into a set of prime factors. We know that 12 can be written as 3 × 4. So, we will find out the largest power of 3 that can divide 200! and the largest power of 4 that can divide 200! and take the LOWER of the two as the largest power of 12 that can divide 200!.
To find out the highest power of 4, since 4 itself is not a prime number, we cannot directly apply the successive division method. We first have to find out the highest power of 2 that can divide 200!. Since two 2's taken together will give us a 4, half the power of 2 will give the highest power of 4 that can divide 200!. We find that 197 is the largest power of 2 that can divide 200!. Half this figure-98-will be the largest power of 4 that can divide 200!.

Since the largest power of 3 and 4 that can divide 200! are 97 an 98 respectively, the smaller of the two, i.e., 97 will be the largest power of 12 that can divide 200! without leaving any remainder.

Arithmetic on Even and Odd Numbers

Addition and subtraction

- even ± even = even
- even ± odd = odd
- odd ± odd = even.

Multiplication

- even × even = even
- even × odd = even
- odd × odd = odd.

Division

- even / odd = even or fraction
- odd / even is never an integer
- odd / odd = odd or fraction
- even / even = even or odd or fraction

Always Keep in Mind

1. The number 1 is neither prime nor composite.

2. 2 is the only even number which is prime.

3. If n is an odd number, then $n(n^2 - 1)$ is divisible by 24.

4. If n is an odd prime number greater than 3, then $(n^2 - 1)$ is divisible by 24.

5. If n is an odd number, then $(2^n + 1)$ is divisible by 3.

6. If n is an even number, then $(2^n - 1)$ is divisible by 3.

7. If n is an odd number, then $(2^{2n} + 1)$ is divisible by 5.

8. If n is an even number, then $(2^{2n} - 1)$ is divisible by 5.

9. If n is even number, then $(2^{2n} - 1)$ is divisible by 15.

10. If n is an odd number, then $(5^{2n} + 1)$ is divisible by 13.

11. If n is an even number, then $(5^{2n} - 1)$ is divisible by 13.

12. If n is any natural number, then $(5^{2n} - 1)$ is divisible by 24.

13. $(x^n + y^n)$ is divisible by $(x + y)$, when n is an odd number.

14. $(x^n - y^n)$ is divisible by $(x + y)$, when n is an even number.

15. $(x^n - y^n)$ is divisible by $(x - y)$, when n is an odd or an even number.

16. The difference between 2 numbers (xy) – (yx) will always be divisible by 9.

17. The square of an odd number when divided by 8 will always give 1 as a remainder.

18. A square of a natural number cannot end with 2, 3, 7, 8 and an odd number of zeros.

19. The square of an odd number is odd and that of an even number is even.

20. Every square number is a multiple of 3 or exceeds a multiple of 3 by unity.

21. Every square number is a multiple of 4 or exceeds a multiple of 4 by unity.

22. If a square number ends in 9, the preceding digit is even.

23. If m and n are two integers, then (m + n)! is divisible by m! n!

24. $(a)^n / (a + 1)$ leaves a remainder of $\begin{cases} a & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$.